TUTORIAL NOTES FOR MATH4220

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1. Green's functions and parabolic equations

Let us recall some useful facts for solving the problems of parabolic equations.

Theorem 1 (Green's identity). *Suppose $u, \partial_t u, \partial_x^2 u \in C([0,\infty)\times\overline{\Omega})$. Then for any $(t,x) \in (0,\infty)\times\Omega$, there holds*

$$u(t,x) = \int_\Omega u(0,x)\Gamma(t,x;0,\xi)d\xi + \int_0^t \int_\Omega \Gamma(t,x;\tau,\xi)[\partial_\tau u(\tau,\xi) - \Delta_\xi u(t,\xi)]d\xi d\tau$$
$$+ \int_0^t \int_{\partial\Omega} \left(\Gamma(t,x;\tau,\xi)\frac{\partial u}{\partial n}(\tau,\xi) - u(\tau,\xi)\frac{\partial \Gamma}{\partial n}(t,x;\tau,\xi)\right) dS_y d\tau,$$

where

$$\Gamma(t,x;\tau,\xi) = [4\pi(t-\tau)]^{-\frac{n}{2}} e^{-\frac{|x-\xi|^2}{4(t-\tau)}} H(t-\tau),$$

$H(z)$ is the Heaviside step function.

For the Dirichlet boundary value problem,

$$\begin{aligned}\partial_t u(t,x) - \Delta_x u(t,x) &= f(t,x), \quad t>0, x\in\Omega,\\ u(0,x) &= u_0(x), \quad x\in\Omega,\\ u(t,x) &= u_b(t,x), \quad t>0, x\in\partial\Omega.\end{aligned}$$

It suffices to find the Green's function $G(t,x;\tau,\xi) = \Gamma(t,x;\tau,\xi) + \Psi(t,x;\tau,\xi)$ where

$$\begin{aligned}-\partial_\tau\Psi(t,x;\tau,\xi) - \Delta_\xi\Psi(t,x;\tau,\xi) &= 0, \quad \tau<t, \xi\in\Omega,\\ \Psi(t,x;t,\xi) &= 0, \quad \xi\in\Omega\\ \Psi(t,x;\tau,\xi) &= -\Gamma(t,x;\tau,\xi), \quad \tau<t, \xi\in\partial\Omega,\end{aligned}$$

then

$$u(t,x) = \int_\Omega u_0(x)G(t,x;0,\xi)d\xi + \int_0^t\int_\Omega G(t,x;\tau,\xi)f(\tau,\xi)d\xi d\tau$$
$$- \int_0^t\int_{\partial\Omega} u_b(t,\xi)\frac{\partial G}{\partial n}(t,x;\tau,\xi)dS_\xi d\tau.$$

In the following, we discuss some examples concerning the Green's function.

Problem 2. Find the Green's function for Dirichlet boundary value problem over the half line

$$\mathbb{R}_+ = \{x\in\mathbb{R} : x>0\}.$$

Solution. The Green's function is

$$G(t,x;\tau,\xi) = [4\pi(t-\tau)]^{-\frac{1}{2}} H(t-\tau)\left(e^{-\frac{|x-\xi|^2}{4(t-\tau)}} - e^{-\frac{|x+\xi|^2}{4(t-\tau)}}\right).$$

Problem 3. Find the Green's function for Dirichlet boundary value problem over the interval

$$I = \{x \in \mathbb{R} : 0 < x < l\}.$$

Solution. The Green's function is

$$G(t, x; \tau, \xi) = \sum_{n=-\infty}^{\infty} [4\pi(t-\tau)]^{-\frac{1}{2}} H(t-\tau) \left(e^{-\frac{|x-\xi-2nl|^2}{4(t-\tau)}} - e^{-\frac{|x+\xi-2nl|^2}{4(t-\tau)}} \right).$$

A Supplementary Problem

Problem 4. Show that for the Neumann boundary value problem,

$$\partial_t u(t, x) - \Delta_x u(t, x) = f(t, x), \quad t > 0, x \in \Omega,$$
$$u(0, x) = u_0(x), \quad x \in \Omega,$$
$$\frac{\partial u}{\partial n}(t, x) = u_b(t, x), \quad t > 0, x \in \partial\Omega.$$

If there exists a Green's function $G(t, x; \tau, \xi) = \Gamma(t, x; \tau, \xi) + \Psi(t, x; \tau, \xi)$ where

$$-\partial_\tau \Psi(t, x; \tau, \xi) - \Delta_\xi \Psi(t, x; \tau, \xi) = 0, \quad \tau < t, \xi \in \Omega,$$
$$\Psi(t, x; t, \xi) = 0, \quad \xi \in \Omega$$
$$\frac{\partial \Psi}{\partial n}(t, x; \tau, \xi) = -\frac{\partial \Gamma}{\partial n}(t, x; \tau, \xi), \quad \tau < t, \xi \in \partial\Omega,$$

then

$$u(t, x) = \int_\Omega u(0, x) G(t, x; 0, \xi) d\xi + \int_0^t \int_\Omega G(t, x; \tau, \xi)[\partial_\tau u(\tau, \xi) - \Delta_\xi u(t, \xi)] d\xi d\tau$$
$$+ \int_0^t \int_{\partial\Omega} G(t, x; \tau, \xi) \frac{\partial u}{\partial n}(\tau, \xi) dS_y d\tau.$$

Moreover, find the Green's function for $\Omega = \mathbb{R}_+ := \{x \in \mathbb{R} : x > 0\}$ and $\Omega = I := \{x \in \mathbb{R} : 0 < x < l\}$.

For more materials, please refer to [1, 2, 3, 4].

References

[1] S. Alinhac, *Hyperbolic partial differential equations*, Universitext, Springer, Dordrecht, 2009.

[2] L. C. Evans, *Partial differential equations*, vol. 19 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 1998.

[3] Q. Han and F. Lin, *Elliptic partial differential equations*, vol. 1 of Courant Lecture Notes in Mathematics, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1997.

[4] W. A. Strauss, *Partial differential equations. An introduction*, John Wiley & Sons, Inc., New York, 1992.

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